Lesson 20  Cofunction Identities

We continue learning more fundamental trigonometric identities. Remember that you are responsible for learning all of these identities as well as their proofs.

Cofunction Identities

1. \( \sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta) \)
2. \( \cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta) \)
3. \( \tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta) \)
4. \( \cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta) \)
5. \( \sec\left(\frac{\pi}{2} - \theta\right) = \csc(\theta) \)
6. \( \csc\left(\frac{\pi}{2} - \theta\right) = \sec(\theta) \)

Before proving them let’s discuss why these are called “cofunction identities.” Recall that the complement of an angle \( \theta \) is the angle \( \frac{\pi}{2} - \theta \) (measured in radians). Identity 1 above can be verbalized as “the cosine of an angle equals the sine of the complement of the angle.” Likewise, identity 2 says that “the sine of an angle equals the cosine of the complement of the angle.” The sine and cosine functions are related by complementary angles in this particular way. We call the sine and cosine functions “cofunctions” because of this relationship. Identities 3 and 4 express that the tangent and cotangent functions are cofunctions, and identities 5 and 6 express that the secant and cosecant functions are cofunctions. This is why the names of three of the trigonometric functions are simply the names of the other three with “co” in front. If you read this paragraph a few times it will give you the key to easily remembering the six identities above.

Proof of 1
We use the subtraction identity for sine:
\[
\sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2}\right)\cos(\theta) - \cos\left(\frac{\pi}{2}\right)\sin(\theta) = 1 \cdot \cos(\theta) - 0 \cdot \sin(\theta) = \cos(\theta).
\]

Proof of 2
We use identity 1 with \( \theta \) replaced by \( \frac{\pi}{2} - \theta \):
\[
\cos\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right) = \sin(\theta).
\]

Note that it is permissible to use previously proven identities to help prove other identities. However, it doesn’t make sense to use identities that haven’t been proven yet.
Proof of 3
We use the tangent/cotangent identities as well as identities 1 and 2 above:

\[ \tan \left( \frac{\pi}{2} - \theta \right) = \frac{\sin \left( \frac{\pi}{2} - \theta \right)}{\cos \left( \frac{\pi}{2} - \theta \right)} = \frac{\cos(\theta)}{\sin(\theta)} = \cot(\theta). \]

Proof of 4
We use identity 3 with \( \theta \) replaced by \( \frac{\pi}{2} - \theta \):

\[ \cot \left( \frac{\pi}{2} - \theta \right) = \tan \left( \frac{\pi}{2} - \left( \frac{\pi}{2} - \theta \right) \right) = \tan(\theta). \]

Proof of 5
We use reciprocal identities as well as identity 2 above:

\[ \sec \left( \frac{\pi}{2} - \theta \right) = \frac{1}{\cos \left( \frac{\pi}{2} - \theta \right)} = \frac{1}{\sin(\theta)} = \csc(\theta). \]

Proof of 6
We use identity 5 with \( \theta \) replaced by \( \frac{\pi}{2} - \theta \):

\[ \csc \left( \frac{\pi}{2} - \theta \right) = \sec \left( \frac{\pi}{2} - \left( \frac{\pi}{2} - \theta \right) \right) = \sec(\theta). \]

That ends the proofs. It is worth noting that there are simpler, more visual proofs in the case when \( \theta \) is an acute angle. Simply consider the following triangle:

![Triangle Diagram]

Using the definitions of the trigonometric functions we have, for example:

\[ \sin \left( \frac{\pi}{2} - \theta \right) = \frac{a}{c} = \cos(\theta). \] The other cofunction identities can be checked similarly.